We have thus shown that for a given separable potential of the form (21) an energy value is found for which the homogeneous equation (9) has an infinite number of solutions. Hence, the corresponding inhomogeneous equation (4) also has an infinite number of solutions.

We point out that the satisfaction of condition (12) for separable potentials leads to the result that the system of integral equations derived from the operator equations [3],

$$T_{12}(z') = t_{12}(z_{12}) + t_{12}(z_{12}) G_0(z') T_{34}(z'),$$
$$T_{34}(z') = t_{34}(z_{34}) + t_{34}(z_{34}) G_0(z') T_{12}(z'),$$

where

$$T_{12}(z') + T_{34}(z') = T(z')$$

and T(z') satisfies Eq. (2), has the same properties as does the original equation (1). The corresponding homogeneous system can have an infinite number of solutions, and the inhomogeneous system can have an infinite number of solutions or be incompatible. The results of this study can be generalized to the scattering problem of five or more bodies under the assumption of pair interactions.

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GENERALIZATION OF SOMMERFELD HEAT-CONDUCTION PROBLEM FOR A RING

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A solution is obtained for the problem of heat conduction in a one-dimensional ring consisting of two sections with different lengths, heat sources, and thermophysical parameters.

The problem of the heat conduction in a ring [1] is an example of a boundary-value problem in which there are no boundary conditions of the first, second, and third kinds modeling the effect of the external medium on the system. In view of the symmetry of the problem, these conditions are replaced by the periodicity condition for the solution. Such "self-closed" systems may serve as mathematical models of different processes of heat and mass transfer [2, 3].

§1. Consider the problem of determining the temperature field in a one-dimensional composite ring, the n sections of which have different lengths, thermophysical parameters, and heat sources. Any of the sections may be regarded as a system interacting with its "environment" – the other sections. The initial temperature distribution in the different sections of the ring is described by different functions and is discontinuous at the contact points, where boundary conditions of the fourth kind are assumed. Since a one-dimensional problem is considered, the shape of the ring is unimportant, as in [1]. A linear coordinate x_i is introduced for each section, $x_i \in (0, l_i)$, i = 1, 2, ..., n. The mathematical formulation of the linear heat-conduction problem for a composite ring takes the form

$$\frac{\partial \tilde{T}_i}{\partial t} - a_i \frac{\partial^2 \tilde{T}_i}{\partial x_i^2} = \tilde{f}_i + \varphi_i \delta(t), \quad t \geqslant 0, \quad x_i \in (0, l_i), \tag{1.1}$$

$$\tilde{T}_i = \Theta(t) T_i(x_i, t), \quad \tilde{f}_i = \Theta(t) f_i(x_i, t), \quad a_i = \lambda_i/(\rho c)_i,$$

$$\varphi_i = \varphi_i(x_i), \quad \delta(t) = \frac{d\Theta}{dt}, \quad \Theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leqslant 0, \end{cases}$$

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$$\tilde{T}_i(x_i, +0) = \varphi_i(x_i), \quad x_i \in (0, l_i), \tag{1.2}$$

$$\tilde{T}_i(l_i - 0, t) = \tilde{T}_{i+1}(+0, t), \quad t > 0 \ (i \neq n), \tag{1.3}$$

$$\tilde{T}_1(+0, t) = \tilde{T}_n(l_n - 0, t), \ t > 0, \tag{1.4}$$

$$\lambda_i \frac{\partial \tilde{T}_i(l_i - 0, t)}{\partial x_i} = \lambda_{i+1} \frac{\partial \tilde{T}_{i+1}(+0, t)}{\partial x_{i+1}}, \quad t > 0 \ (i \neq n), \tag{1.5}$$

$$\lambda_1 \frac{\partial \tilde{T}_1(+0, t)}{\partial x_1} = \lambda_n \frac{\partial \tilde{T}_n(l_n - 0, t)}{\partial x_n}, \quad t > 0. \tag{1.6}$$

The system (1.1) together with the conditions (1.2)-(1.6) constitute a generalized Cauchy problem [4], the solution of which is constructed in two stages. The first stage is to solve the first boundary-value problem (1.1)-(1.4); for this purpose, "matching" functions are introduced:

$$\tilde{\mu}_i(t) = \tilde{T}_i(+0, t), \ t > 0 \ \ (i = 1, 2, \ldots, n). \tag{1.7}$$

As a result, the functions $\tilde{T}_i(x_i, t)$ are found; these include differential and integral expressions in $\tilde{\mu}_i(t)$. The second stage is to use the conditions (1.5) and (1.6). Substituting $\tilde{T}_i(x_i, t)$ into Eqs. (1.5) and (1.6), a system of n integral equations of convolution type in the n unknown functions $\mu_i(t)$ is obtained; this system may be solved using a Laplace transform. So as to obtain explicit expressions, consideration is limited to the case $n = 2$ in the present work. It is expedient to superimpose the coordinate origins of the axes x_i $(i = 1, 2)$, which results in sign reversal of one of the thermal fluxes in Eqs. (1.5) and (1.6) and redefinition of the matching functions:

$$\tilde{\mu}_1(t) = \tilde{T}_i(+0, t), \ \ \tilde{\mu}_2(t) = \tilde{T}_i(l_i - 0, t) \quad (i = 1, 2).$$

The Green-function method [5] gives

$$\tilde{T}_i(x_i, t) = \int_0^t \int_0^{l_i} \tilde{G}_i(x_i, \xi, t - \tau)\, \tilde{\omega}_i(\xi, \tau)\, d\xi d\tau + \tilde{M}(x_i, t), \tag{1.8}$$

where

$$\tilde{G}_i(x_i, \xi, t) = \frac{2\Theta(t)}{l_i} \sum_{n=1}^{\infty} E_n^{(i)}(t) \sin\left(\frac{n\pi}{l_i} x_i\right) \sin\left(\frac{n\pi}{l_i} \xi\right)$$

is the Green function of the first boundary-value problem for the segment,

$$\tilde{\omega}_i(x_i, t) = \tilde{f}_i(x_i, t) + \varphi_i(x_i)\delta(t) - \frac{d\tilde{M}}{dt},$$

$$\tilde{M}(x_i, t) = \tilde{\mu}_1(t) + \frac{x_i}{l_i}(\tilde{\mu}_2(t) - \tilde{\mu}_1(t)), \tag{1.9}$$

$$E_n^{(i)}(t) = \exp\left[-\left(\frac{n\pi}{l_i}\right)^2 a_i t\right].$$

Substituting Eq. (1.8) into Eqs. (1.5) and (1.6) yields a system of two integral equations of convolution type in the two unknown functions $\mu_j(t)$ $(j = 1, 2)$. If a Laplace transform is now applied to this system, the result is

$$\sum_{j=1}^{2} \bar{a}_{kj}(p)\, \bar{\mu}_j(p) = \bar{c}_k(p) \quad (k = 1, 2). \tag{1.10}$$

In Eq. (1.10), $\bar{c}_k(p)$ and $\bar{a}_{kj}(p)$ are Laplace transforms of certain functions of the time; $\bar{\mu}_j(p)$ are the transforms of the functions $\tilde{\mu}_j(t)$; p is the Laplace-transform parameter (omitted below).

§2. Thus, the boundary-value problem reduces to the algebraic system (1.10). Solving system (1.10) and performing an inverse Laplace transform yields $\tilde{\mu}_j(t)$, substitution of which into Eqs. (1.9) and (1.8) completes the solution. When $\mathrm{Re}\, p \geq \sigma_0 > 0$, $\mathrm{Det}\,||\bar{a}_{kj}(p)|| \neq 0$; this allows a unique solution of Eq. (1.10) to be obtained in the form

$$\bar{\mu}_j = \bar{\Phi}_j^{(1)} + \bar{\Phi}_j^{(2)}, \tag{2.1}$$

where

$$\bar{\Phi}_j^{(i)} = \bar{N}_+ \bar{F}_-^{(i)} + (-1)^{j+1} \bar{N}_- \bar{F}_+^{(i)}, \tag{2.2}$$

$$\bar{N}_\pm = \bar{N}_\pm^{(1)} + \bar{N}_\pm^{(2)}, \ \ \bar{N}_\pm^{(i)} = \varepsilon_i (\mathrm{ch}\, \delta_i \sqrt{p} \pm 1)/\bar{D}\sqrt{p}\, \mathrm{sh}\, \delta_i \sqrt{p}, \tag{2.3}$$

$$\bar{D} = \varepsilon_1^2 + \varepsilon_2^2 - 2\varepsilon_1\varepsilon_2((1-\operatorname{ch}\delta_1\sqrt{p}\operatorname{ch}\delta_2\sqrt{p})/\operatorname{sh}\delta_1\sqrt{p}\operatorname{sh}\delta_2\sqrt{p}), \tag{2.4}$$

$$\bar{F}_-^{(i)} = 2\sum_{k=1}^{\infty}(p + \beta_{2k-1}^{(i)})^{-1}\bar{J}_{2k-1}^{(i)}, \tag{2.5}$$

$$\bar{F}_+^{(i)} = 2\sum_{k=1}^{\infty}(p + \beta_{2k}^{(i)})^{-1}\bar{J}_{2k}^{(i)}, \tag{2.6}$$

$$\bar{J}_n^{(i)} = \frac{\lambda_i}{l_i}\int_0^{l_i}[\bar{f}_i(\xi, p) + \varphi_i(\xi)]\left(\frac{n\pi}{l_i}\right)\sin\left(\frac{n\pi}{l_i}\xi\right)d\xi, \quad \varepsilon_i = \sqrt{\lambda_i(\rho c)_i}, \quad \delta_i = l_i/\sqrt{a_i}, \quad \beta_n^{(i)} = (n\pi/l_i)^2 a_i. \tag{2.7}$$

The inverse transform is trivial for all the functions except $\bar{N}_\pm$, which are transformed using residues [6]:

$$N_+(t) = 2\left(\frac{\lambda_1}{l_1} + \frac{\lambda_2}{l_2}\right)(\varepsilon_1 + \varepsilon_2)^{-2} + \sum_{n=1}^{\infty}A_+(n)E_n(t) +$$

$$+ 2(\varepsilon_1\delta_1 + \varepsilon_2\delta_2)^{-1}\left\{\sum_{n=1}^{\infty}\delta_{nn'}E_n^{(1)}(t) + \sum_{m=1}^{\infty}\delta_{mm'}E_m^{(2)}(t)\right\}, \tag{2.8}$$

$$N_-(t) = \sum_{n=1}^{\infty}A_-(n)E_n(t). \tag{2.9}$$

In Eqs. (2.8) and (2.9),

$$A_\pm(n) = \frac{\varepsilon_1^{-1}\sin\delta_1\alpha_n(\cos\delta_2\alpha_n \pm 1) + \varepsilon_2^{-1}\sin\delta_2\alpha_n(\cos\delta_1\alpha_n \pm 1)}{[\delta_1(\sin\delta_1\alpha_n)^{-1} - \delta_2(\sin\delta_2\alpha_n)^{-1}](\cos\delta_2\alpha_n - \cos\delta_1\alpha_n)},$$

$$E_n(t) = \exp(-\alpha_n^2 t), \quad \delta_{nn'} = \begin{cases}1, & n = n'\\ 0, & n \neq n',\end{cases}$$

where α_n are the roots of the equation

$$(\varepsilon_1^2 + \varepsilon_2^2)\sin\delta_1\alpha\sin\delta_2\alpha + 2\varepsilon_1\varepsilon_2(1 - \cos\delta_1\alpha\cos\delta_2\alpha) = 0.$$

The expression in curly brackets in Eq. (2.8) is nonzero only in the case $n'/m' = \delta_2/\delta_1$.

§3. In the limiting case when one of the sections of the composite ring contracts to a point, the solution obtained transforms to a solution of the Sommerfeld problem [1]. The corresponding calculations are rather cumbersome, and so the discussion will be confined to the limiting transition for the temperature at the contact point of the sections. For consistency with [1], let

$$\bar{f}_i(x_i, t) = 0, \quad \varphi_i(x_i) = \delta\left(x_i - \frac{l_i}{2}\right), \quad \delta_1 = \delta_2 = \delta, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon. \tag{3.1}$$

Substituting Eq. (3.1) into Eq. (2.7) yields

$$\bar{J}_n^{(i)} = \begin{cases}0, & n = 2k, \ k = 1, 2, \ldots\\ (-1)^{k+1}\dfrac{\lambda_i}{l_i}\dfrac{2k-1}{l_i}\pi, & n = 2k-1.\end{cases} \tag{3.2}$$

Then

$$\bar{F}_+^{(i)} = 0, \quad \bar{\Phi}_1^{(i)} = \bar{\Phi}_2^{(i)}, \quad \bar{\mu}_1 = \bar{\mu}_2 = \bar{\mu}. \tag{3.3}$$

From Eqs. (2.1)-(2.4) it is found that

$$\bar{\mu} = \frac{\pi}{2lp}\left[\sum_{k=1}^{\infty}(-1)^{k+1}(2k-1)(p + \beta_{2k-1})^{-1}\right]\left[\sum_{k=1}^{\infty}(p + \beta_{2k-1})^{-1}\right]^{-1},$$

which may be reduced, using [7], to the form

$$\bar{\mu} = \frac{1}{2\sqrt{ap}}\,\frac{1}{\operatorname{sh}\dfrac{\delta\sqrt{p}}{2}}. \tag{3.4}$$

Using [8], the following expression is obtained for the original of Eq. (3.4):

$$\tilde{\mu}(t)=\frac{\Theta(t)}{l}\Theta_3\left(\frac{1}{2},\ \frac{4at}{l^2}\right)=$$

$$=\Theta(t)\left[\frac{1}{l}+\frac{2}{l}\sum_{k=1}^{\infty}(-1)^k\exp\left(-\frac{4k^2\pi^2}{l^2}at\right)\right].$$

For $l=1$ this result agrees with the expression for $\tilde{\mu}(t)$ for $u(\pm 1/2, t)$ in [1].

§4. Consider the behavior of $\tilde{\mu}_j(t)$ at the initial moment of time. The well-known relation of [6] will be used:

$$\tilde{\mu}_j(+0)=\lim_{t\to+0}\tilde{\mu}_j(t)=\lim_{p\to\infty}p\bar{\mu}_j(p). \tag{4.1}$$

It may be shown that there is no contribution to $\tilde{\mu}_j(+0)$ from finite $\tilde{f}_i(x_i, t)$; therefore, setting $\tilde{f}_i(x_i, t)=0$, consider the particular case of a linear initial temperature distribution

$$\varphi_i(x_i)=\varphi_{i0}+(\varphi_{il}-\varphi_{i0})\frac{x_i}{l_i},\quad \varphi_{i0}=\varphi_i(0),\ \varphi_{il}=\varphi_i(l_i). \tag{4.2}$$

After simple transformations, Eqs. (2.1)-(2.7) yield

$$\tilde{\mu}_1(+0)=\frac{K_\varepsilon\varphi_{10}+\varphi_{20}}{1+K_\varepsilon},\quad \tilde{\mu}_2(+0)=\frac{K_\varepsilon\varphi_{1l}+\varphi_{2l}}{1+K_\varepsilon}, \tag{4.3}$$

where the notation of [8] is used: $K_\varepsilon=\varepsilon_1/\varepsilon_2$. Thus, in the case of a linear initial temperature profile, the initial discontinuity of the matching function $\tilde{\mu}_j(t)$, i.e., $h_j=\tilde{\mu}_j(+0)-\tilde{\mu}_j(-0)=\tilde{\mu}_j(+0)$, is determined solely by the boundary values.

The case of constant but different initial temperatures in the sections of the ring may be considered by setting $\varphi_{i0}=\varphi_{il}=T_{i0}$ in Eq. (4.2). Then Eq. (4.3) gives

$$\tilde{\mu}_1(+0)=\tilde{\mu}_2(+0)=\frac{K_\varepsilon T_{10}+T_{20}}{1+K_\varepsilon}. \tag{4.4}$$

If the sections have the same thermophysical parameters ($\varepsilon_1=\varepsilon_2$, $K_\varepsilon=1$), Eq. (4.4) yields

$$\tilde{\mu}_1(+0)=\tilde{\mu}_2(+0)=\frac{T_{10}+T_{20}}{2}$$

in agreement with the problem on the contact of two half-spaces with different initial temperatures [8].

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